

Chapter 4A: Exponential Functions Outline

Lesson 1: Integer Exponents



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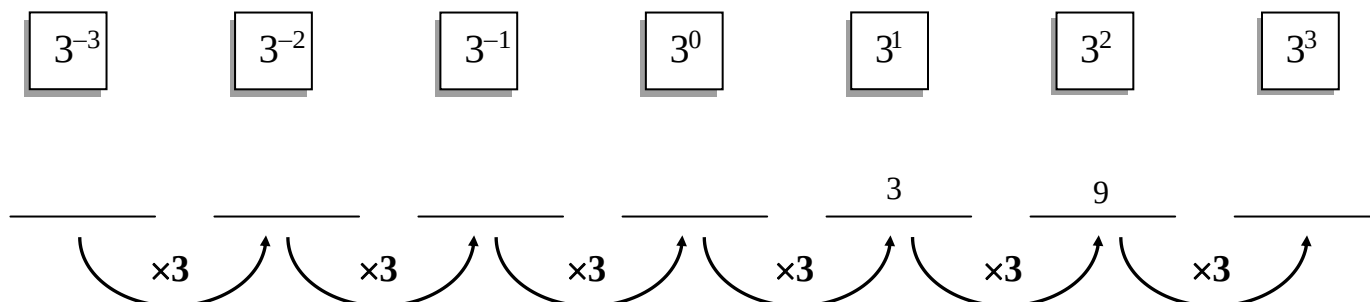


LESSON 1

INTEGER EXPONENTS

We just finished our review of linear functions. Linear functions are those that grow by equal differences for equal intervals. In this unit we will concentrate on exponential functions which grow by equal factors for equal intervals. To understand exponential functions, we first need to understand exponents.

Exercise #1: The following sequence shows powers of 3 by repeatedly multiplying by 3. Fill in the missing blanks.



This pattern can be duplicated for any **base** raised to any **integer exponent**. Because of this we can now define positive, negative, and zero exponents in terms of multiplying the number 1 repeatedly or dividing the number 1 repeatedly.

INTEGER EXPONENT DEFINITIONS

If n is any positive integer then:

$$1. \ b^n = 1 \cdot \underbrace{b \cdot b \cdot b \cdot \dots \cdot b \cdot b}_{n\text{-times}}$$

$$2. \ b^0 = 1$$

$$3. \ b^{-n} = \frac{1}{\underbrace{b \cdot b \cdot b \cdot \dots \cdot b \cdot b}_{n\text{-times}}} = \frac{1}{b^n}$$

Exercise #2: Given the exponential function $f(x) = 20(2)^x$ evaluate each of the following without using your calculator. Show the calculations that lead to your final answer.

(a) $f(2)$

(b) $f(0)$

(c) $f(-2)$

There are many basic **exponent properties or laws** that are critically important and that can be investigated using integer exponent examples. Two of the very important ones we will see next.

Exercise #3: For each of the following, write the product as a single exponential expression. Write (a) and (b) as extended products first (if necessary).

(a) $2^3 \cdot 2^4$

(b) $2^6 \cdot 2^2$

(c) $2^m \cdot 2^n$

It's clear why the exponent law that you generalized in part (c) works for positive integer exponents. But, does it also make sense within the context of our negative exponents?

Exercise #4: Consider now the product $2^3 \cdot 2^{-1}$.

(a) Use the exponent law found in Exercise 3(c) to write this as a single exponential expression.

(b) Evaluate $2^3 \cdot 2^{-1}$ by first rewriting 2^3 and 2^{-1} and then simplifying.

(c) Do your answers from (a) and (b) support the extension of the **Addition Property of Exponents** to negative powers as well? Explain.

Let's look at another important exponent property.

Exercise #5: For each of the following, write the exponential expression in the form 3^x . Write (a) and (b) as extended products first (if necessary).

(a) $(3^2)^3$

(b) $(3^4)^2$

(c) $(3^m)^n$

Again, let's look at how the **Product Property of Exponents** still holds for negative exponents.

Exercise #6: Consider the expression $(3^{-2})^4$. Show this expression is equivalent to 3^{-8} by first rewriting 3^{-2} in fraction form.

Now let's go the other way around. For each of the following, determine the integer value of n that satisfies the equation.

Exercise #7: $2^n = 8$

(Rewrite both sides with the same base – here both sides should be written as a power of 2)

When the bases are the same on both sides of the equal sign, the exponents are equal.

Exercise #8:

a) $6^n = 1296$

b) $4^n = \frac{1}{64}$

c) $8^n = \frac{1}{512}$

d) $3^{x-2} = 27$

e) $4^4 = 2^{3x-1}$

f) $81^x = 9^{x+1}$

g) $4^{x+1} = 8^x$

h) $\frac{1^x}{5} = 125^{3x-4}$

i) $6^{4x+3} = \frac{1}{216}^{2x}$

j) $2^{x^2} = 4^x$

LESSON 1 HOMEWORK

INTEGER EXPONENTS

FLUENCY

1. Write each of the following exponential expressions without the use of exponents such as we did in lesson *Exercise #1*.

(a)

2^{-3}	_____	) ×2
2^{-2}	_____	
2^{-1}	_____	) ×2
2^0	_____	
2^1	2	) ×2
2^2	4	
2^3	_____	) ×2

(b)

5^{-3}	_____	) ×5
5^{-2}	_____	
5^{-1}	_____	) ×5
5^0	_____	
5^1	5	) ×5
5^2	25	
5^3	_____	) ×5

2. Now let's go the other way around. For each of the following, determine the integer value of n that satisfies the equation. The first is done for you.

(a) $2^n = \frac{1}{8}$

(b) $4^n = 16$

(c) $3^n = \frac{1}{81}$

(d) $7^n = 1$

$2^n = \frac{1}{2^3}$

$2^n = 2^{-3}$

$n = -3$

$$(e) 5^n = \frac{1}{25}$$

$$(f) 10^n = \frac{1}{10,000}$$

$$(g) 13^n = 1$$

$$(h) 2^n = \frac{1}{32}$$

3. Use the **Addition Property of Exponents** to simplify each expression. Then, find a final numerical answer *without* using your calculator.

$$(a) 2^{-5} \cdot 2^3 \cdot 2^4$$

$$(b) 5^3 \cdot 5^7 \cdot 5^{-10}$$

$$(c) 10^3 \cdot 10^{-7} \cdot 10^2$$

4. Use the **Product Property of Exponents** to simplify each exponential expression. You do not need to find a final numerical answer.

$$(a) (2^3)^4$$

$$(b) (3^{-2})^2$$

$$(c) \left((5^2)^{-4}\right)^{-2}$$

5. The exponential expression $\left(\frac{1}{8}\right)^4$ is equivalent to which of the following? Explain your choice.

$$(1) 4^{-8}$$

$$(3) 8^{-2}$$

$$(2) 2^{-12}$$

$$(4) 32^{-1}$$

REASONING

6. How can you use the fact that $25^2 = 625$ to show that $5^{-4} = \frac{1}{625}$? Explain your process of thinking.
7. We've extended the two fundamental exponent properties to negative as well as positive integers. What would happen if we extended the **Product Exponent Property** to a fractional exponent like $\frac{1}{2}$? Let's play around with that idea.
- (a) Use the **Product Property of Exponents** to justify that $\left(9^{\frac{1}{2}}\right)^2 = 9$.
- (b) What other number can you square that results in 9?

Lesson 1 Homework Answer Key

1. (a) $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$ (b) $\frac{1}{125}, \frac{1}{25}, \frac{1}{5}, 1, 5, 25, 125$
2. (b) 2, (c) -4, (d) 0, (e) -2, (f) -3, (g) 0, (h) -5
3. (a) $2^2 = 4$, (b) $5^0 = 1$, (c) $10^{-2} = \frac{1}{100}$
4. (a) 2^{12} , (b) 3^{-4} , (c) 5^{16}
5. (a) 2^{-12} . Because $\left(\frac{1}{8}\right)^4 = (2^{-3})^4$
6. $5^{-4} = \frac{1}{5^4} = \frac{1}{(5^2)^2} = \frac{1}{25^2} = \frac{1}{625}$
7. (a) $\left(9^{\frac{1}{2}}\right)^2 = 9^{\left(\frac{1}{2} \cdot 2\right)} = 9^1 = 9$ (b) 3. Therefore since $\left(9^{\frac{1}{2}}\right)^2 = 9$ and $(3)^2 = 9 \dots 9^{\frac{1}{2}} = 3$

LESSON 2

RATIONAL EXPONENTS

When you first learned about exponents, they were always **positive integers**, and just represented **repeated multiplication**. And then we had to go and introduce **negative exponents**, which really just represent **repeated division**. Today we will introduce **rational (or fractional) exponents** and extend your exponential knowledge that much further.

Exercise #1: Recall the **Product Property of Exponents** and use it to rewrite each of the following as a simplified exponential expression. There is no need to find a final numerical value.

(a) $(2^3)^4$

(b) $(5^{-2})^5$

(c) $(3^7)^0$

(d) $\left((4^2)^{-2}\right)^2$

We will now use the Product Property to extend our understanding of exponents to include **unit fraction** exponents (those of the form $\frac{1}{n}$ where n is a positive integer).

Exercise #2: Consider the expression $16^{\frac{1}{2}}$.

(a) Apply the Product Property to simplify $(16^{\frac{1}{2}})^2$. What other number squared yields 16?

(b) You can now say that $16^{\frac{1}{2}}$ is equivalent to what more familiar quantity?

Conclusion: An exponent of $1/2$ is equivalent to the square root of a number.

Write it again:

Exercise #3: Test the equivalence of the $\frac{1}{2}$ exponent to the square root by using your calculator to evaluate each of the following. Be careful in how you enter each expression.

(a) $25^{\frac{1}{2}} =$

(b) $81^{\frac{1}{2}} =$

(c) $100^{\frac{1}{2}} =$

We can extend this now to all levels of roots, that is square roots, cubic roots, fourth roots, etcetera.

UNIT FRACTION EXPONENTS

For n given as a positive integer:

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

Exercise #4: Rewrite each of the following using roots instead of fractional exponents. Then, if necessary, evaluate using your calculator to guess and check to find the roots (don't use the generic root function). Check with your calculator.

(a) $125^{\frac{1}{3}}$

(b) $16^{\frac{1}{4}}$

(c) $9^{-\frac{1}{2}}$

(d) $32^{-\frac{1}{5}}$

We can now combine traditional integer powers with unit fractions in order to interpret any exponent that is a **rational number**, i.e. the **ratio of two integers**. The next exercise will illustrate the thinking. Remember, we want our exponent properties to be consistent with the structure of the expression.

Exercise #5: Let's think about the expression $4^{\frac{3}{2}}$.

(a) Fill in the missing blank and then evaluate this expression:

$$4^{\frac{3}{2}} = (\quad)^{\frac{1}{2}}$$

(b) Fill in the missing blank and then evaluate this expression:

$$4^{\frac{3}{2}} = (\quad)^3$$

(c) Verify both (a) and (b) using your calculator.

(d) Evaluate $27^{\frac{2}{3}}$ without your calculator. Show your thinking. Verify with your calculator.

RATIONAL EXPONENT CONNECTION TO ROOTS

For the rational number $\frac{m}{n}$ we define $b^{\frac{m}{n}}$ to be: $\sqrt[n]{b^m}$ or $(\sqrt[n]{b})^m$.

Exercise #6: Evaluate each of the following exponential expressions involving rational exponents without the use of your calculator. Show your work. Then, check your final answers with the calculator.

(a) $16^{\frac{3}{4}}$

(b) $25^{\frac{3}{2}}$

(c) $8^{-\frac{2}{3}}$

LESSON 2 HOMEWORK

RATIONAL EXPONENTS

FLUENCY

1. Rewrite the following as equivalent roots and then evaluate as many as possible **without your calculator**.

(a) $36^{\frac{1}{2}}$

(b) $27^{\frac{1}{3}}$

(c) $32^{\frac{1}{5}}$

(d) $100^{-\frac{1}{2}}$

(e) $625^{\frac{1}{4}}$

(f) $49^{\frac{1}{2}}$

(g) $81^{-\frac{1}{4}}$

(h) $343^{\frac{1}{3}}$

2. Evaluate each of the following by considering the root and power indicated by the exponent. Do as many as possible **without your calculator**.

(a) $8^{\frac{2}{3}}$

(b) $4^{\frac{3}{2}}$

(c) $16^{-\frac{3}{4}}$

(d) $81^{\frac{5}{4}}$

(e) $4^{-\frac{5}{2}}$

(f) $128^{\frac{3}{7}}$

(g) $625^{\frac{3}{4}}$

(h) $243^{\frac{3}{5}}$

3. Given the function $f(x) = 5(x+4)^{\frac{3}{2}}$, which of the following represents its y -intercept?

(1) 40

(3) 4

(2) 20

(4) 30

4. Which of the following is equivalent to $x^{-1/2}$?

(1) $-\frac{1}{2}x$

(3) $\frac{1}{\sqrt{x}}$

(2) $-\sqrt{x}$

(4) $-\frac{1}{2x}$

5. Written without fractional or negative exponents, $x^{-3/2}$ is equal to

(1) $-\frac{3x}{2}$

(3) $\frac{1}{\sqrt{x^3}}$

(2) $\frac{1}{\sqrt[3]{x^2}}$

(4) $-\frac{1}{\sqrt{x}}$

6. Which of the following is *not* equivalent to $16^{3/2}$?

(1) $\sqrt{4096}$

(3) 64

(2) 8^3

(4) $\sqrt{16^3}$

REASONING

7. Marlene claims that the square root of a cube root is a sixth root? Is she correct? To start, try rewriting the expression below in terms of fractional exponents. Then apply the **Product Property of Exponents**.

$$\sqrt{\sqrt[3]{a}}$$

8. We should know that $\sqrt[3]{8} = 2$. To see how this is equivalent to $8^{1/3} = 2$ we can solve the equation $8^n = 2$. To do this, we can rewrite the equation as:

$$(2^3)^n = 2^1$$

How can we now use this equation to see that $8^{1/3} = 2$?

Lesson 2 Homework Answer Key

1. (a) 6, (b) 3, (c) 2, (d) $\frac{1}{10}$, (e) 5, (f) 7, (g) $\frac{1}{3}$, (h) 7

2. (a) 4, (b) 8, (c) $\frac{1}{8}$, (d) 243, (e) $\frac{1}{32}$, (f) 8, (g) 125, (h) 27

3. (1) 4. (3) 5. (2) 6. (2)

7. Marlene is correct. Re-write the expression using fractional exponents, then simplify.

8. $2^3 = 8$. Use this to get equal bases, then solve.

LESSON 3

EXPONENTIAL FUNCTION BASICS

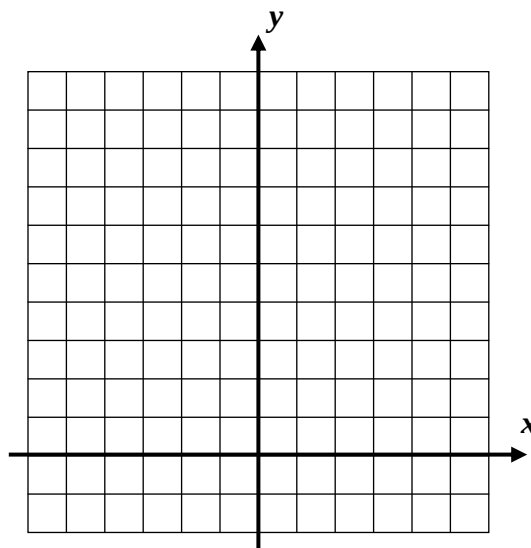
You studied exponential functions extensively in Common Core Algebra I. Today's lesson will review many of the basic components of their graphs and behavior. Exponential functions, those whose exponents are variable, are extremely important in mathematics, science, and engineering.

BASIC EXPONENTIAL FUNCTIONS

$$y = b^x \text{ where } b > 0 \text{ and } b \neq 1$$

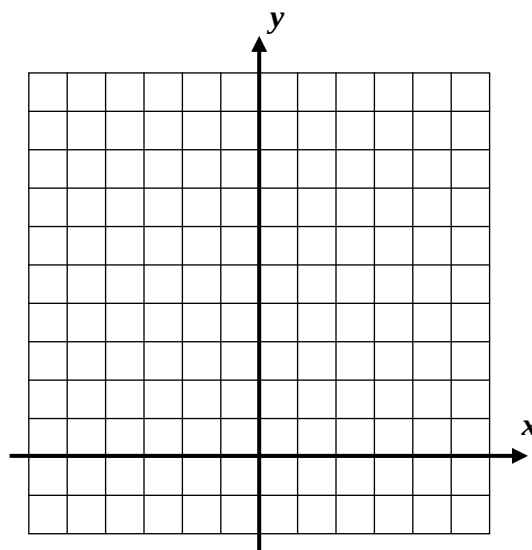
Exercise #1: Consider the function $y = 2^x$. Fill in the table below without using your calculator and then sketch the graph on the grid provided.

x	$y = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	



Exercise #2: Now consider the function $y = \left(\frac{1}{2}\right)^x$. Using your calculator to help you, fill out the table below and sketch the graph on the axes provided.

x	$y = \left(\frac{1}{2}\right)^x$
-3	
-2	
-1	
0	
1	
2	
3	



Exercise #3: Based on the graphs and behavior you saw in *Exercises #1 and #2*, state the domain and range for an exponential function of the form $y = b^x$.

Domain (input set):

Range (output set):

Does a graph of an exponential function contain an asymptote? How can you tell?

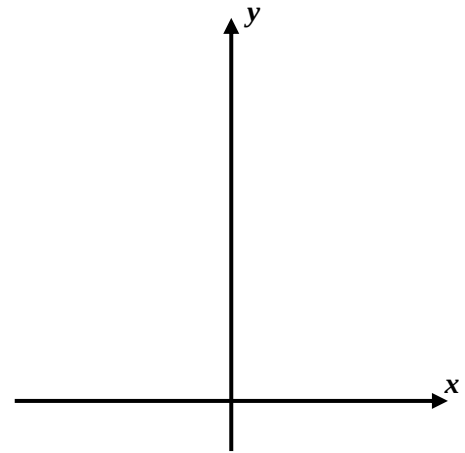
Exercise #4: Are exponential functions one-to-one? How can you tell? What does this tell you about their inverses?

Exercise #5: Now consider the function $y = 7(3)^x$.

(a) Determine the y -intercept of this function algebraically. Justify your answer.

(b) Does the exponential function increase or decrease? Explain your choice.

(c) Create a rough sketch of this function, labeling its y -intercept.



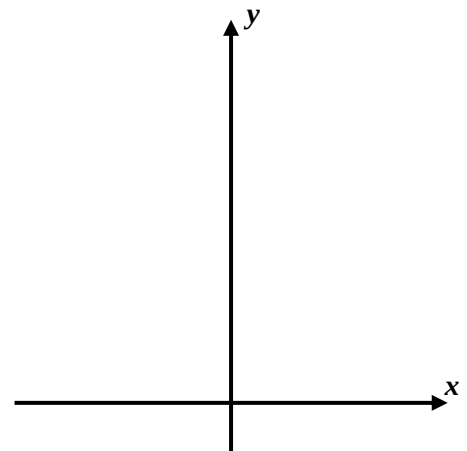
Exercise #6: Consider the function $y = \left(\frac{1}{3}\right)^x + 4$.

(a) How does this function's graph compare to that of $y = \left(\frac{1}{3}\right)^x$? What does adding 4 do to a function's graph?

(b) Determine this graph's y -intercept algebraically. Justify your answer.

(c) Create a rough sketch of this function, labeling its y -intercept.

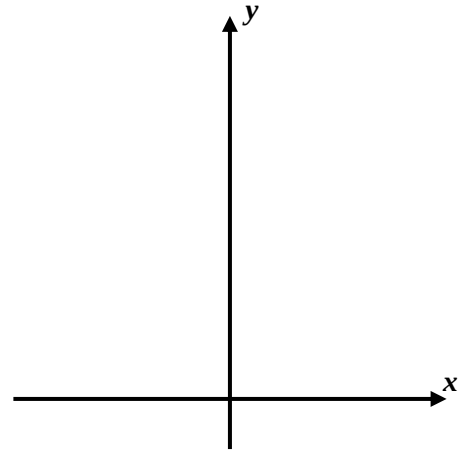
(d) Where is the asymptote on this graph? Draw it in.



Exercise 7: Consider the function $y = 5^x - 6$.

(a) How does the graph of this function compare to the graph of $y = 5^x$? Explain in words.

(b) Sketch the function on the graph provided.



Vertical Shifts: When something is being added to the basic exponential function $y = b^x$ the graph is being shifted _____. When something is *subtracted* from the basic exponential function $y = b^x$, the graph shifts _____.

LESSON 3 HOMEWORK
EXPONENTIAL FUNCTION BASICS

FLUENCY

1. Which of the following represents an exponential function?

(1) $y = 3x - 7$

(3) $y = 3(7)^x$

(2) $y = 7x^3$

(4) $y = 3x^2 + 7$

2. If $f(x) = 6(9)^x$ then $f\left(\frac{1}{2}\right) = ?$ (Remember what we just learned about fractional exponents and do without a calculator.)

(1) $\frac{7}{2}$

(3) 27

(2) 18

(4) $\frac{15}{2}$

3. If $h(x) = 3^x$ and $g(x) = 5x - 7$ then $h(g(2)) =$

(1) 18

(3) 38

(2) 12

(4) 27

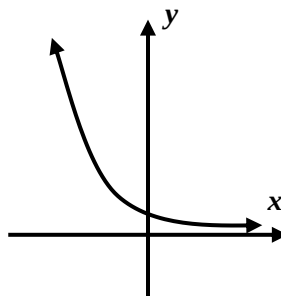
4. Which of the following equations could describe the graph shown below?

(1) $y = x^2 + 1$

(3) $y = -2x + 1$

(2) $y = \left(\frac{2}{3}\right)^x$

(4) $y = 4^x$



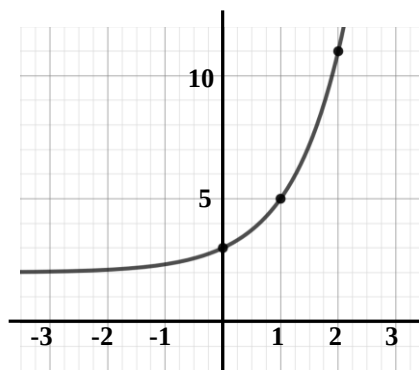
5. Which of the following equations represents the graph shown?

(1) $y = 5^x$

(3) $y = \left(\frac{1}{2}\right)^x + 2$

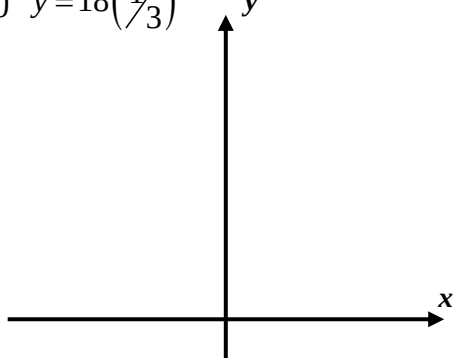
(2) $y = 4^x + 1$

(4) $y = 3^x + 2$

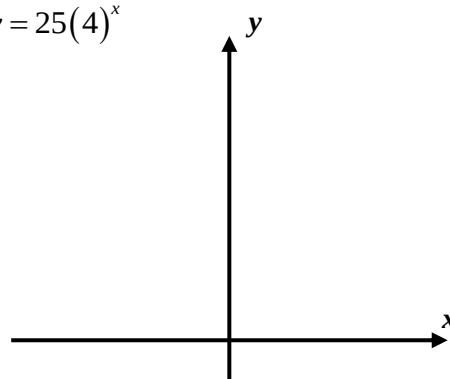


6. Sketch graphs of the equations shown below on the axes given. Label the y-intercepts of each graph.

(a) $y = 18\left(\frac{1}{3}\right)^x$



(b) $y = 25(4)^x$



APPLICATION

7. The Fahrenheit temperature of a cup of coffee, F , starts at a temperature of 185°F . It cools down according to the exponential function $F(m) = 113\left(\frac{1}{2}\right)^{m/20} + 72$, where m is the number minutes it has been cooling.
- (a) How do you interpret the statement that $F(60) = 86$?
- (b) Determine the temperature of the coffee after one day using your calculator. What do you think this temperature represents about the physical situation?

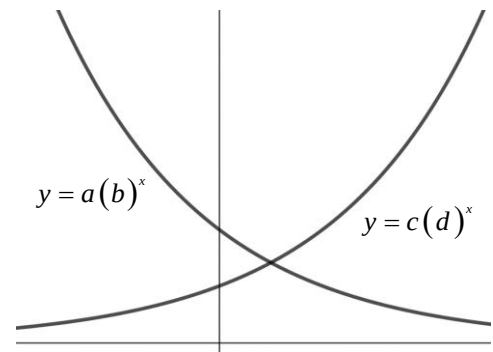
REASONING

8. The graph below shows two exponential functions, with real number constants a , b , c , and d . Given the graphs, only one pair of the constants shown below could be equal in value. Determine which pair could be equal and explain your reasoning.

b and d

a and b

a and c



9. Explain why the equation below can have no real solutions. If you need to, graph both sides of the equation using your calculator to visualize the reason.

$$3^x + 5 = 2$$

Lesson 3 Homework Answer Key

1. (3) 2. (2) 3. (4) 4. (2) 5. (4) 6. Check in your calculator or [desmos.com/calculator](https://www.desmos.com/calculator)
7. (a) After 60 minutes, the temp is 86°F, (b) 72°F. This is likely room temperature.
8. a and c. (b is between 0 & 1, d is larger than 1; a & b aren't related in that way)
9. $3^x = -3$. There is no value of x that would yield -3

LESSON 4

FINDING EQUATIONS OF EXPONENTIAL FUNCTIONS

One of the skills that you acquired in Common Core Algebra I was the ability to write equations of exponential functions if you had information about the starting value and base (multiplier or growth constant). Let's review a very basic problem.

Exercise #1: An exponential function of the form $f(x) = a(b)^x$ is presented in the table below. Determine the values of a and b and explain your reasoning.

x	0	1	2	3
$f(x)$	5	15	45	135

$a =$ _____

$b =$ _____

Final Equation: _____

Explanation:

Finding an exponential equation becomes much more challenging if we do not have output values for inputs that are increasing by unit values (increasing by 1 unit at a time). Let's start with a basic problem.

Exercise #2: For an exponential function of the form $f(x) = a(b)^x$, it is known that $f(0) = 8$ and $f(3) = 1000$.

(a) Use the fact that $f(0) = 8$ to determine the value of a . Show your thinking.

(b) Use your answer from part (a) and the fact that $f(3) = 1000$ to set up an equation to solve for b . You will solve for b in part (c).

(c) Solve for the value of b using properties of exponents.

(d) Determine the value of $f(2)$

Exercise #3: An exponential function exists such that $f(4) = 3$ and $f(6) = 48$, which of the following must be the value of its base? Explain or illustrate your thinking.

(1) $b = 16$

(3) $b = 6$

(2) $b = 2$

(4) $b = 4$

Now, let's work with the most generic type of problem. Just like with lines, **any two (non-vertically aligned) points will uniquely determine the equation of an exponential function.**

Exercise #4: An exponential function of the form $y = a(b)^x$ passes through the points $(2, 36)$ and $(5, 121.5)$.

(a) By substituting these two points into the general form of the exponential, create a **system of equations** in the constants a and b .

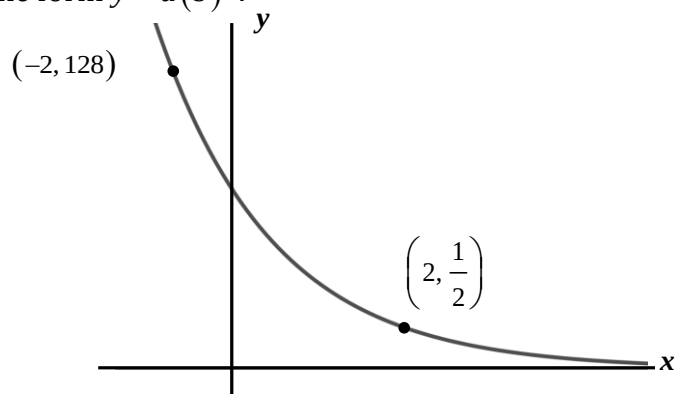
(b) Divide these two equations to eliminate the constant a . Recall that when dividing to like bases, you subtract their exponents.

(c) Solve the resulting equation from (b) for the base, b .

(d) Use your value from (c) to determine the value of a . State the final equation.

Let's now get some practice on this with a decreasing exponential function.

Exercise #5: Find the equation of the exponential function shown graphed below. Be careful in terms of your exponent manipulation. State your final answer in the form $y = a(b)^x$.



Exercise #6: A bacterial colony is growing at an exponential rate. It is known that after 4 hours, its population is at 98 bacteria and after 9 hours it is 189 bacteria. Determine an equation in $y = a(b)^x$ form that models the population, y , as a function of the number of hours, x . At what percent rate is the population growing per hour?

LESSON 4 HOMEWORK

FINDING EQUATIONS OF EXPONENTIAL FUNCTIONS

FLUENCY

1. For each of the following coordinate pairs, find the equation of the exponential function, in the form $y = a(b)^x$ that passes through the pair. Show the work that you use to arrive at your answer.

(a) $(0, 10)$ and $(3, 80)$

(b) $(0, 180)$ and $(2, 80)$

2. For each of the following coordinate pairs, find the equation of the exponential function, in the form $y = a(b)^x$ that passes through the pair. Show the work that you use to arrive at your answer.

(a) $(2, 192)$ and $(5, 12288)$

(b) $(1, 192)$ and $(5, 60.75)$

3. Each of the previous problems had values of a and b that were rational numbers. They do not need not be. Find the equation for an exponential function that passes through the points $(2, 14)$ and $(7, 205)$ in $y = a(b)^x$ form. When you find the value of b do not round your answer before you find a . Then, find both to the nearest *hundredth* and give the final equation. Check to see if the points fall on the curve.

APPLICATIONS

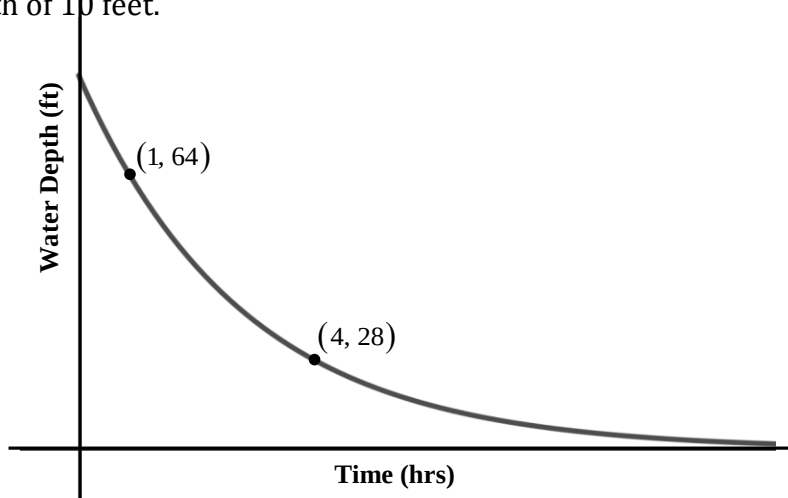
4. A population of koi goldfish in a pond was measured over time. In the year 2002, the population was recorded as 380 and in 2006 it was 517. Given that y is the population of fish and x is the number of years *since* 2000, do the following:

- (a) Represent the information in this problem as two coordinate points.
- (b) Determine a linear function in the form $y = mx + b$ that passes through these two points. Don't round the linear parameters (m and b).

- (c) Determine an exponential function of the form $y = a(b)^x$ that passes through these two points. Round b to the nearest hundredth and a to the nearest *tenth*.

- (d) Which model predicts a larger population of fish in the year 2000? Justify your work.

5. Engineers are draining a water reservoir until its depth is only 10 feet. The depth decreases exponentially as shown in the graph below. The engineers measure the depth after 1 hour to be 64 feet and after 4 hours to be 28 feet. Develop an exponential equation in $y = a(b)^x$ to predict the depth as a function of hours draining. Round a to the nearest integer and b to the nearest hundredth. Then, graph the horizontal line $y = 10$ and find its intersection to determine the time, to the nearest tenth of an hour, when the reservoir will reach a depth of 10 feet.



Lesson 4 Homework Answer Key

1. (a) $y = 10(2)^x$ (b) $y = 180\left(\frac{2}{3}\right)^x$ 2. (a) $y = 12(4)^x$ (b) $y = 256\left(\frac{3}{4}\right)^x$

3. $y = 4.78(1.71)^x$. Both points are close, but not quite. Due to rounding.

4. (a) (2, 380), (6, 517) (b) $y = 34.25x + 311.5$ (c) $y = 325.8(1.08)^x$

(d) For each model the year 2000 represents a time value of zero, . So, we are just comparing the y-intercepts. The linear has a value of 311.5 and the exponential a value of 325.8. So the exponential has the higher population in the year 2000.

5. $y = 84(0.76)^x$

LESSON 5

THE METHOD OF COMMON BASES

There are very few algebraic techniques that **do not involve technology** to solve equations that contain **exponential expressions**. In this lesson we will look at one of the few, known as **The Method of Common Bases**.

Exercise #1: Solve each of the following simple exponential equations by writing each side of the equation using a **common base**.

(a) $2^x = 16$

(b) $3^x = 27$

(c) $5^x = \frac{1}{25}$

(d) $16^x = 4$

In each of these cases, even the last, more challenging one, we could manipulate the right-hand side of the equation so that it shared a **common base** with the left-hand side of the equation. We can exploit this fact by manipulating both sides so that they have a common base. First, though, we need to review an exponent law.

Exercise #2: Simplify each of the following exponential expressions.

(a) $(2^3)^x$

(b) $(3^2)^{4x}$

(c) $(5^{-1})^{3x-7}$

(d) $(4^{-3})^{1-x^2}$

Exercise #3: Solve each of the following equations by finding a common base for each side.

(a) $8^x = 32$

(b) $9^{2x+1} = 27$

(c) $125^x = \left(\frac{1}{25}\right)^{4-x}$

Exercise #4: Which of the following represents the solution set to the equation $2^{x^2-3} = 64$?

(1) $\{\pm 3\}$

(3) $\{\pm\sqrt{11}\}$

(2) $\{0, 3\}$

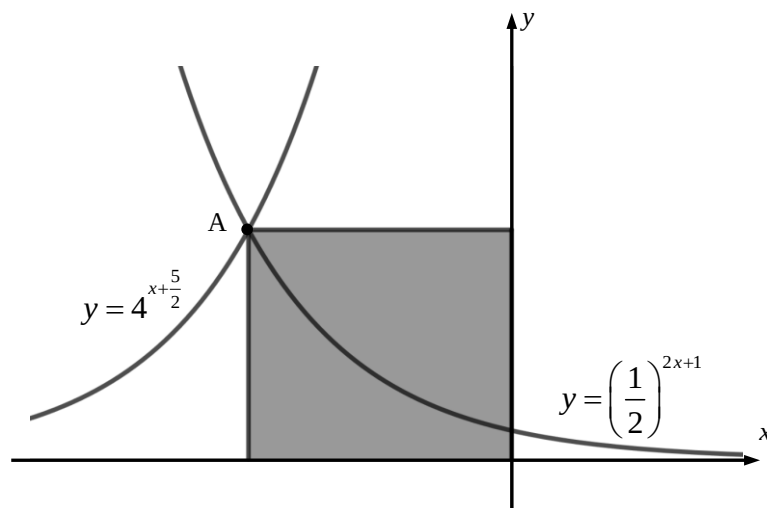
(4) $\{\pm\sqrt{35}\}$

This technique can be used in any situation where all bases involved can be written with a common base. In a practical sense, this is rather rare. Yet, these types of algebraic manipulations help us see the **structure** in **exponential expressions**. Try to tackle the next, more challenging, problem.

Exercise #5: Two exponential curves, $y = 4^{x+\frac{5}{2}}$ and $y = \left(\frac{1}{2}\right)^{2x+1}$ are shown below. They intersect at point

A. A rectangle has one vertex at the origin and the other at A as shown. We want to find its area.

(a) Fundamentally, what do we need to know about a rectangle to find its area?



(b) How would knowing the coordinates of point A help us find the area?

(c) Find the area of the rectangle algebraically using the Method of Common Bases. Show your work carefully.

Exercise #6: At what x coordinate will the graph of $y = 25^{x-a}$ intersect the graph of $y = \left(\frac{1}{125}\right)^{3x+1}$? Show the work that leads to your choice.

(1) $x = \frac{5a-1}{3}$

(3) $x = \frac{-2a+1}{5}$

(2) $x = \frac{2a-3}{11}$

(4) $x = \frac{5a+3}{2}$

LESSON 5 HOMEWORK
THE METHOD OF COMMON BASES

FLUENCY

1. Solve each of the following exponential equations using the Method of Common Bases. Each equation will result in a linear equation with one solution. Check your answers.

(a) $3^{2x-5} = 9$

(b) $2^{3x+7} = 16$

(c) $5^{4x-5} = \frac{1}{125}$

(d) $8^x = 4^{2x+1}$

(e) $216^{x-2} = \left(\frac{1}{1296}\right)^{3x-2}$

(f) $\left(\frac{1}{25}\right)^{x+15} = 3125^{\frac{3}{5}x-1}$

2. *Algebraically* determine the intersection point of the two exponential functions shown below. Recall that most systems of equations are solved by substitution.

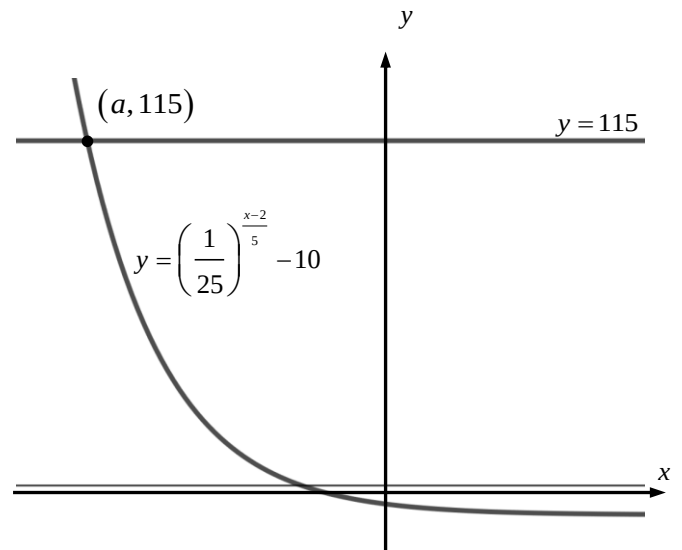
$$y = 8^{x-1} \quad \text{and} \quad y = 4^{2x-3}$$

3. *Algebraically* determine the **zeroes** of the exponential function $f(x) = 2^{2x-9} - 32$. Recall that the reason it is known as a zero is because the **output is zero**.

APPLICATIONS

4. One hundred must be raised to what power in order to be equal to a million cubed? Solve this problem using the Method of Common Bases. Show the algebra you do to find your solution.

5. The exponential function $y = \left(\frac{1}{25}\right)^{\frac{x-2}{5}} - 10$ is shown graphed along with the horizontal line $y = 115$. Their intersection point is $(a, 115)$. Use the Method of Common Bases to find the value of a . Show your work.



REASONING

- 6 The Method of Common Bases works because exponential functions are one-to-one, i.e. if the outputs are the same, then the inputs must also be the same. This is what allows us to say that if $2^x = 2^3$, then x must be equal to 3. But it doesn't always work out so easily.

If $x^2 = 5^2$, can we say that x must be 5? Could it be anything else? Why does this not work out as easily as the exponential case?

Lesson 5 Homework Answer Key

1. (a) $7/2$ (b) -1 (c) $\frac{1}{2}$ (d) -2 (e) $14/15$ (f) -5
2. $(3, 64)$ 3. $x = 7$ 4. 9 5. -5.5

6. In this case, x could be either 5 or -5 . It does not work out as easily as with exponential functions because the squaring function is not one-to-one.

LESSON 6

EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY

Exponential functions are very important in modeling a variety of real world phenomena because certain things either increase or decrease by **fixed percentages** over given units of time. You looked at this in Common Core Algebra I and in this lesson we will review much of what you saw.

Exercise #1: Suppose that you deposit money into a savings account that receives 5% interest per year on the amount of money that is in the account for that year. Assume that you deposit \$400 into the account initially.

- (a) How much will the savings account increase by over the course of the year?
- (b) How much money is in the account at the end of the year?
- (c) By what single number could you have multiplied the \$400 by in order to calculate your answer in part (b)?
- (d) Using your answer from part (c), determine the amount of money in the account after 2 and 10 years. Round all answers to the nearest cent when needed.
- (e) Give an equation for the amount in the savings account $S(t)$ as a function of the number of years since the \$400 was invested.
- (f) Using a table on your calculator determine, to the nearest year, how long it will take for the initial investment of \$400 to double. Provide evidence to support your answer.

The thinking process from *Exercise #1* can be generalized to any situation where a quantity is increased by a fixed percentage over a fixed interval of time. This pattern is summarized below:

INCREASING EXPONENTIAL MODELS

If quantity Q is known to increase by a fixed percentage p , in decimal form, then Q can be modeled by

$$Q(t) = Q_0(1 + p)^t$$

where Q_0 represents the amount of Q present at $t = 0$ and t represents time.

Exercise #2: Which of the following gives the savings S in an account if \$250 was invested at an interest rate of 3% per year?

(1) $S = 250(4)^t$ (3) $S = (1.03)^t + 250$

(2) $S = 250(1.03)^t$ (4) $S = 250(1.3)^t$

Decreasing exponentials are developed in the same way, but have the percent subtracted, rather than added, to the base of 100%. Just remember, you are ultimately multiplying by the percent of the original that you will have after the time period elapses.

Exercise #3: State the multiplier (base) you would need to multiply by in order to decrease a quantity by the given percent listed.

(a) 10%

(b) 2%

(c) 25%

(d) 0.5%

DECREASING EXPONENTIAL MODELS

If quantity Q is known to decrease by a fixed percentage p , in decimal form, then Q can be modeled by

$$Q(t) = Q_0(1 - p)^t$$

where Q_0 represents the amount of Q present at $t = 0$ and t represents time.

Exercise #4: If the population of a town is decreasing by 4% per year and started with 12,500 residents, which of the following is its projected population in 10 years? Show the exponential model you use to solve this problem.

(1) 9,230

(3) 18,503

(2) 76

(4) 8,310

Exercise #5: The stock price of Windpower Inc. is increasing at a rate of 4% per week. Its initial value was \$20 per share. On the other hand, the stock price in GerbilEnergy is crashing (losing value) at a rate of 11% per week. If its price was \$120 per share when Windpower was at \$20, after how many weeks will the stock prices be the same? Model both stock prices using exponential functions. Then, find when the stock prices will be equal graphically. Draw a well labeled graph to justify your solution.

LESSON 6 HOMEWORK
EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY

APPLICATIONS

1. If \$130 is invested in a savings account that earns 4% interest per year, which of the following is closest to the amount in the account at the end of 10 years?

(1) \$218 (3) \$168

(2) \$192 (4) \$324

2. A population of 50 fruit flies is increasing at a rate of 6% per day. Which of the following is closest to the number of days it will take for the fruit fly population to double?

(1) 18 (3) 12

(2) 6 (4) 28

3. If a radioactive substance is quickly decaying at a rate of 13% per hour approximately how much of a 200 pound sample remains after one day?

(1) 7.1 pounds (3) 25.6 pounds

(2) 2.3 pounds (4) 15.6 pounds

4. A population of llamas stranded on a dessert island is decreasing due to a food shortage by 6% per year. If the population of llamas started out at 350, how many are left on the island 10 years later?

(1) 257 (3) 102

(2) 58 (4) 189

5. Which of the following equations would model a population with an initial size of 625 that is growing at an annual rate of 8.5%?

(1) $P = 625(8.5)^t$ (3) $P = 1.085^t + 625$

(2) $P = 625(1.085)^t$ (4) $P = 8.5t^2 + 625$

6. The acceleration of an object falling through the air will decrease at a rate of 15% per second due to air resistance. If the initial acceleration due to gravity is 9.8 meters per second per second, which of the following equations best models the acceleration t seconds after the object begins falling?

(1) $a = 15 - 9.8t^2$ (3) $a = 9.8(1.15)^t$

(2) $a = \frac{9.8}{15t}$ (4) $a = 9.8(0.85)^t$

7. Red Hook has a population of 6,200 people and is growing at a rate of 8% per year. Rhinebeck has a population of 8,750 and is growing at a rate of 6% per year. In how many years, to the nearest year, will Red Hook have a greater population than Rhinebeck? Show the equation or inequality you are solving and solve it graphically.

8. A warm glass of water, initially at 120 degrees Fahrenheit, is placed in a refrigerator at 34 degrees Fahrenheit and its temperature is seen to decrease according to the exponential function

$$T(h) = 86(0.83)^h + 34$$

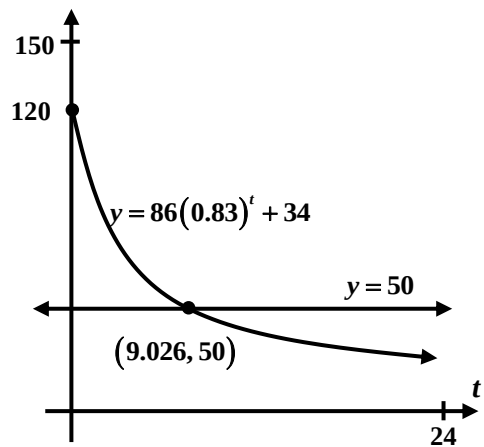
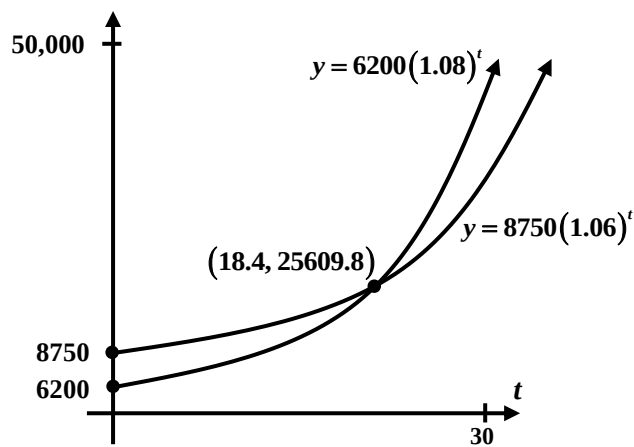
- (a) Verify that the temperature starts at 120 degrees Fahrenheit by evaluating $T(0)$.
- (b) Using your calculator, sketch a graph of T below for all values of h on the interval $0 \leq h \leq 24$. Be sure to label your y -axis and y -intercept.
- (c) After how many hours will the temperature be at 50 degrees Fahrenheit? State your answer to the nearest *hundredth* of an hour. Illustrate your answer on the graph you drew in (b).

REASONING

9. Percents combine in strange ways that don't seem to make sense at first. It would seem that if a population grows by 5% per year for 10 years, then it should grow in total by 50% over a decade. But this isn't true. Start with a population of 100. If it grows at 5% per year for 10 years, what is its population after 10 years? What percent growth does this represent?

Lesson 6 Homework Answer Key

1. (2) 2. (3) 3. (1) 4. (4) 5. (2) 6. (4)
7. 18 years (graph below, left)
8. (a) 120 (b) graph below, right (c) 9.03 hours
9. 163.63%



LESSON 7

MINDFUL MANIPULATION OF PERCENTS

Percents and phenomena that grow at a constant percent rate can be challenging, to say the least. This is due to the fact that, unlike linear phenomena, the growth rate indicates a constant multiplier effect instead of a constant additive effect (linear). Because constant percent growth is so common in everyday life (not to mention in science, business, and other fields), it's good to be able to **mindfully manipulate percents**.

Exercise 1: A population of wombats is growing at a constant percent rate. If the population on January 1st is 1027 and a year later is 1079, what is its yearly percent growth rate to the nearest tenth of a percent?

Exercise 2: Now let's try to determine what the percent growth in wombat population will be over a decade of time. We will assume that the rounded perfect increase found in Exercise 1 continues for the next decade.

(a) After 10 years, what will we have multiplied the original population by, rounded to the nearest hundredth? Show calculation.

(b) Using your answer from (a), what is the decade percent growth rate?

Exercise 3: Let's stick with our wombats from Exercise 1. Assuming their growth rate is constant over time, what is their monthly growth rate to the nearest tenth of a percent? Assume a constant sized month.

Exercise 4: If a population was growing at a constant rate of 22% every 5 years, then what is its percent growth rate over a 2 year time span? Round to the nearest tenth of a percent.

(a) First, give an expression that will calculate the single year (or yearly) percent growth rate based on the fact that the population grew 22% in 5 years.

(b) Now use this expression to calculate the percent growth over 2 years.

Exercise 5: World oil reserves (the amount of oil unused in the ground) are depleting at a constant rate of 2% per year. We would like to determine what percent decline will be over the next 20 years based on this 2% yearly decline.

(a) Write and evaluate an expression for what we would multiply the initial amount of oil by after 20 years.

(b) Use your answer to (a) to determine the percent decline after 20 years. Be careful! Round to the nearest percent.

Exercise 6: A radioactive substance's half-life is the amount of time needed for half (or 50%) of the substance to decay. We have a radioactive substance with a half-life of 20 years.

(a) What percent of the substance would be radioactive after 40 years?

(b) What percent of the substance would be radioactive after only 10 years? Round your answer to the nearest tenth of a percent.

(c) What percent of the substance would be radioactive after only 5 years? Round your answer to the nearest tenth of a percent.

LESSON 7 HOMEWORK
MINDFUL MANIPULATION OF PERCENTS

1. A quantity is growing at a constant 3% yearly rate. Which of the following would be its percent growth after 15 years?

(1) 45%

(3) 56%

(2) 52%

(4) 63%

2. If a credit card company charges 13.5% yearly interest, which of the following calculations would be used in the process of calculating the monthly interest rate?

(1) $\frac{0.135}{12}$

(3) $(1.135)^{12}$

(2) $\frac{1.135}{12}$

(4) $(1.135)^{1/12}$

3. The county debt is growing at an annual rate of 3.5%. What percent rate is it growing at per 2 years? Per 5 years? Per decade? Show calculations that lead to each answer. Round each to the nearest tenth of a percent.

4.) A population of llamas is growing at a constant yearly rate of 6%. At what rate is the llama population growing per month? Please assume all months are equally sized and that there are 12 of these per year. Round to the nearest tenth of a percent.

5. Shana is trying to increase the number of calories she burns by 5% per day. By what percent is she trying to increase per week? Round to the nearest tenth of a percent.

6. If a bank account doubles in size every 5 years, then by what percent does it grow after only 3 years? Round to the nearest tenth of a percent. Hint: First write an expression that would calculate its growth rate after a single year.

7. An object's speed decreases by 5% for each minute that it is slowing down. Which of the following is closest to the percent that its speed will decrease over half-an hour?

(1) 21%

(3) 48%

(2) 79%

(4) 150%

8. Over the last 10 years, the price of corn has decreased by 25% per bushel.

(a) Assuming a steady percent decrease, by what percent does it decrease each year? Round to the nearest tenth of a percent.

(b) Assuming this percent continues, by what percent will the price of corn decrease by after 50 years? Show the calculation that leads to your answer. Round to the nearest percent.

Lesson 7 Homework Answer Key

1. (3) 2. (4)
3. 2 years: 7.1%; 5 years: 18.8%; decade: 41.1%
4. 0.5% per month
5. 40.7%
6. 51.6%
7. (2)
8. 2.8%
9. 76%